Stanford University Mathematics Camp (SUMaC) 2021 Admissions Exam

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- ❖ Solve as many of the following problems as you can. Your work on these problems together with your grades in school, teacher recommendations, and answers to questions on the application form are all used to evaluate your SUMaC application. Although SUMaC is very selective with a competitive applicant pool, correct answers on every problem are not required for admission.
- ❖ There is no time limit for this exam other than the application deadline of March 10.
- ❖ Feel free to report partial progress toward a solution, in the event you are unable to solve a problem completely.
- ❖ You will need to create a separate document with your solutions and notes. This document may be typed or handwritten, as long as the final document you upload is legible for our review. Please include clear, detailed explanations on ***all of your answers***; numerical answers or formulas with no explanation are not useful for evaluating your application.
- ❖ None of these problems require a calculator or computer, and they are all designed so that they can be done without computational tools.
- ❖ You are expected to do your own work without the use of any outside source (books, internet search, etc). If you recognize one of the problems from another source, or if you receive any assistance, please indicate this on your solution.
- ❖ ***Please do not share these problems or your solutions with anyone.***

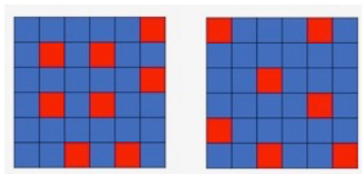
1. Find all positive integers x and y that satisfy $x^2 - y^2 = 2021$? Are there any integers x and y (positive or negative) such that $x^3 - y^3 = 2021$? Explain.

2. Let $p(x)$ be a polynomial with real coefficients such that *the product of all six roots of* $p(x)$ *is negative*. Show that if the degree of $p(x)$ is 6, then $p(x)$ has at least one positive root.

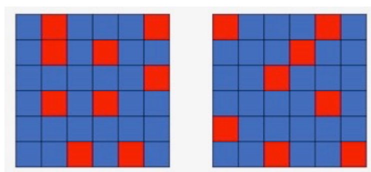
3. The squares in an $n \times n$ grid are colored red and blue with the condition

(1) *No two red squares touch either adjacently or diagonally.*

For example, these two grids with $n = 6$ satisfy condition (1):



And these two grids *do not* satisfy condition (1):



The one on the left has two adjacent red squares, and the one on the right has red squares that touch each other diagonally.

a. Consider the additional condition

(2) *there is exactly one red square in each row, and there is exactly one red square in each column*

For $n = 4$, is there a grid that meets both conditions (1) and (2)?

b. For $n = 5$, how many different ways can the grid be colored satisfying both (1) and (2)?

c. Now consider the condition:

(3) *there are exactly two red squares in each row, and there are exactly two red squares in each column.*

Show that if $n = 7$ then conditions (1) and (3) cannot both be satisfied.

d. Is it possible to satisfy both (1) and (3) if $n = 8$?

e. For $n = 9$, how many distinct grids are there that satisfy both properties (1) and (3)?

f. Now consider the following condition:

(4) *there are exactly three red squares in each row, and there are exactly three red squares in each column.*

What is the smallest value of n such there is a grid that satisfies conditions (1) and (4)?

4. Consider the first quadrant of the Cartesian plane as an infinite checkerboard where each square is labeled (i, j) for non-negative integers i and j.

   a. Suppose you have a game piece called *Duke* which can move from square (a, b) to either square $(a + b, b)$ or to square $(a, a + b)$. Suppose Duke starts at $(1,1)$. Can you characterize the possible squares (i, j) which Duke can travel to in finitely many moves?

   b. Suppose Duke starts at $(6,15)$, to which squares (i, j) can Duke travel?

   c. Suppose you have a game piece called *Duchess* which moves in coordination with another piece of the same kind. If there are two Duchesses on the board, one is allowed to move from (a, b) to $(a + m, b + n)$ whenever m and n have occurred as coordinates of positions that one or the other of the two Duchesses have already occupied. In particular, if points of the form (m, y_1) or (x_1, m), as well as (n, y_2) or (x_2, n), have been visited by either of the Duchesses during the course of the game, then either Duchess may move from (a, b) to $(a + m, b + n)$. Suppose the two Duchesses start at (4, 0) and (0, 5). What are all the possible squares that can be reached after finitely many moves?

5. The imaginary Sea of Le has many (more than four) islands, and travel between them may only take place on bridges. Each pair of islands is connected by exactly one bridge, and no two bridges have the same length. The bridges cross over and under each other in a complex configuration, so the lengths of the bridges are unrelated to the direct distance between the islands which they connect.

   Adi, Bai and Casey are each planning tours that *visit each island exactly once*, though the three of them do not necessarily start on the same island, and none of them return to their own starting island.

   - Casey carefully plans the journey before starting out, finding the best starting island and a route that visits each island along a route that *minimizes the sum total of the lengths* of the bridges traveled.
   - Adi, though, starts on a random island and plans a journey one island at a time. At each island along the way, Adi selects the *shortest* bridge from that island to an island not yet visited.
   - Bai, who prefers a scenic route, also starts on a random island and, at each island along the way, Bai selects the *longest* bridge from that island to an island not yet visited.

   For all three questions below, either show that the proposal is possible by constructing an example or prove why it is not possible.

   a. Is it possible that the total length of Casey's journey over bridges is *strictly less* than Adi's?

   b. Is it possible that all three journeys (Adi's, Bai's, and Casey's) have the same total length?

   c. Is it possible that the total length of Bai's journey is strictly less than Adi's?

6. A non-empty finite set S of positive integers, all strictly greater than 1, has the following two properties:

   (1) If p is prime and there is a multiple of p in S, then there are at least two distinct multiples of p in S.

   (2) If p and q are distinct primes and S contains a multiple of p and a multiple of q, then S has at least one multiple of pq.

   Show that S can be partitioned into three disjoint sets S_1, S_2, and S_3, such that for any prime p, if one of these three sets has a multiple of p, then there is a multiple of p in one of the other two sets.